NON-MEASURABILITY, IMPRECISE CREDENCES, AND IMPRECISE CHANCES

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1. **Introduction**

Orthodox Bayesianism models an agent's epistemic state with a probability function. Each proposition to which the agent has a doxastic attitude is assigned a real number in the unit interval. But a heterodox probabilistic epistemology has also been developed in which each proposition to which the agent has a doxastic attitude is assigned a *range* of real numbers. Orthodox Bayesianism requires agents to have precise credences, while heterodox probabilistic epistemology allows agents to have *imprecise* credences.

This paper offers a new motivation for imprecise credences, one based on the mathematical phenomenon of non-measurable sets. Given natural constraints, not all propositions can receive precise credences. So if precise credences are the only credences allowed, some propositions will be left out. Given analogous constraints, not all propositions can receive precise chances. So if precise chances are the only chances allowed, some propositions will be left out. And leaving propositions out in this way poses myriad difficulties. But if imprecise credences and imprecise chances are allowed, then no propositions need be left out. The framework of imprecise credences and imprecise chances thus has a major, heretofore unappreciated advantage.

Here's our plan: In §2 we explain the mathematics of non-measurable sets. In §3 we argue that if credences must be precise then non-measurability poses severe and widespread problems. In §4 we argue that imprecise credences offer a compelling approach to non-measurable phenomena, thus supporting the rational permissibility of imprecise credences. In §5 we discuss non-measurable chances in more detail and argue for a generalization of the Principal Principle. Although we argue for imprecise credences, in §6 we argue against the standard interpretation of imprecise credences as sets of precise probabilities. Precise credences—even collectively—cannot adequately deal with non-measurability. In §7 we argue that non-measurability offers natural rejoinders to prominent critics of imprecise credences. Non-measurability even turns the tables on some of the critics' arguments against imprecise credences, who by their own lights should welcome such credences. In §8 we explore the prospects for rejecting our reasoning by appealing to cognitive limitations. In §9 we explore some ways of either questioning the existence of non-measurable sets or their application to epistemology.

2. The mathematics of non-measurable sets

Orthodox probability theory is spectacularly flexible. It offers continuum-many values—all the real numbers between 0 and 1. But there are striking mathematical results showing that—given some natural constraints—not all propositions can receive real-valued probabilities.

One famous result emerges from the constraint that probabilities be rotationally symmetric. Suppose that the probabilities for a spinner are fair, in that any rotation of a given set of points must have the same probability as that set of points. For example, since the spinner is certain to land somewhere on a circle, each quadrant of the circle must have the same probability: $\frac{1}{4}$. But some other sets don't work out as neatly.

There are sets of points that cannot receive receive a precise probability. Here's the classic proof (Vitali 1905): Partition the points on the circle into equivalence classes of points that are rational fractions of the circumference away from each other. So the top point of the circle is in the same equivalence class as the bottom point, the right point, and countably many other points. (See Figure 1.) Now take one point from each of the equivalence classes and consider that set of points.[1] Any rational rotation of that set of points will map to a disjoint set of points, of which there are countably many. Moreover, the union of all of those sets is the entire circle. If any of the sets receive a probability, then symmetry requires that each receive the same probability and countable additivity requires that the sum of their probabilities be 1. But that's impossible—if each set receives probability 0 then the sum

[1] This demonstration thus employs the axiom of choice. More about it later.

of their probabilities is 0, and if each set receives probability greater than 0 then the sum of their probabilities is infinite. Either way, the sum is not 1. So none of the sets can receive a probability.

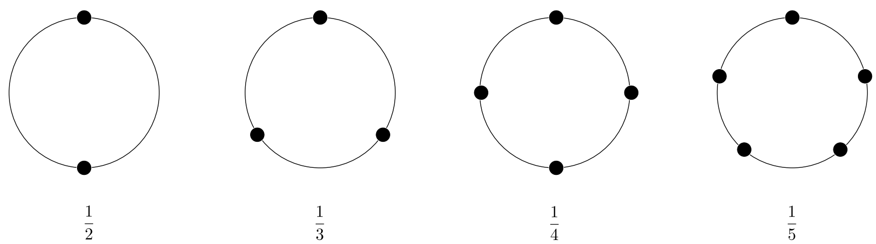

FIGURE 1. These points, together with the points picked out by smaller and smaller fractions, constitute one such equivalence class.

In the mathematics of probability theory, troublesome sets that cannot be given a probability assignment consistently with operative constraints are known as *non-measurable*. We emphasize that it is not that such sets cannot receive a probability, *period*. It's that such sets cannot receive a probability *relative to certain constraints*. (And in some cases the constraints do not entail that any particular set fails to receive a probability, but only that there is a set of sets that cannot all simultaneously receive probabilities.) Non-measurability is a well-known phenomenon in mathematics, and the mathematics of probability theory is structured in light of it. Mathematically, probabilities aren't just the numbers assigned in conformity to the standard axioms. A probability space is constituted by a triple, $\langle \Omega, \mathcal{F}, P \rangle$. Ω is the set on which the probability space is based. $\mathcal{F}$ is a σ-algebra of subsets of Ω that will receive numerical probability assignments. And P is a numerical assignment to each element of $\mathcal{F}$ in a way that conforms to the standard axioms. It's crucial that not all subsets of Ω need be included in $\mathcal{F}$. The whole point of using $\mathcal{F}$ instead of just using the power set of Ω is to be able to leave out some subsets of Ω. This is important because some subsets of Ω may be non-measurable relative to operative constraints. We can't have the mathematics of probability theory crashing every time there are non-measurable sets in the offing. By limiting numerical probabilities to σ-algebras we can have well-behaved probabilities for well-behaved sets and ignore ill-behaved sets entirely.

One might wonder whether we could escape non-measurability (and thus sets not receiving probabilities) by dropping symmetry constraints. Interestingly, even if one does so things are far from plain sailing. Stanislaw Ulam (1930) proved a striking result. Let X be a set of cardinality equal to $\aleph_1$, and let countably additive probabilities be defined over subsets of X.[2] If there is any uncountable subset of X that receives non-zero probability even though each of its elements receives zero probability, then those probabilities are not defined over all subsets of X. This result has pertinent consequences, though they present some technical complications. To simplify things, for now we'll assume that the continuum hypothesis is true. That is, we'll assume that there is no set whose cardinality is strictly between that of the natural numbers and that of the continuum, i.e. that $\aleph_1 = \mathfrak{c}$. We'll revisit this assumption later. Given that assumption, any real-valued probability assignment to all sets of the spinner's landing points must be extremely biased: there is some countable set of points on which it concentrates probability 1, despite there being uncountably many candidate landing points. Any region lacking one of these special points would be guaranteed to have 0 probability. Such a bias would preclude not only a perfectly fair spinner, but any spinner with a continuous probability distribution.[3]

Non-measurable sets are left out of σ-algebras and thus do not receive probabilities. Nonetheless, even non-measurable sets can be analyzed probabilistically. According to orthodox probabilistic mathematics, every non-measurable set has an *inner measure* and an *outer measure.* The inner measure of a set is the upper bound of the measures of its measurable subsets, and the outer measure of a set is the lower bound of the measures of its measurable supersets. We may think of a set's inner measure as approximating its probability 'from below' and its outer measure as approximating its probability 'from above'. If a set's inner and outer measures differ, it is non-measurable. Nonetheless, we may think of inner and outer measures as providing bounds for the probability of a non-measurable set. To speak a bit loosely, a non-measurable set cannot have a particular size, but it's definitely at least as big as any of its subsets and it's definitely no bigger than any of its supersets. The inner measure of any set is guaranteed to be at least 0 (and can be more), and the outer measure of any set is guaranteed to be no more than 1 (and can be less). Non-measurable sets do not receive probabilities, but inner and outer measures provide some probabilistic information about them. Thus non-measurable sets aren't all

[2]The natural numbers have cardinality $\aleph_0$, and $\aleph_1$ is the smallest cardinality strictly greater than that. Note that Ulam's result also applies to many cardinalities other than $\aleph_1$, though we won't be concerned with other cardinalities here.

[3]This is a case in which any given set can receive a probability consistent with operative restraints, but not all sets can simultaneously.

the same probabilistically. Non-measurable sets are troublesome, but not entirely anarchic.

The non-measurable sets from the earlier example have inner measure 0 and (depending on how they are constructed) may have outer measure of any positive number up to 1. Consider the extreme case in which the outer measure is 1 and the points in the set form a dense cloud throughout the circle. Some basic set-theoretic operations can then yield a non-measurable set with intermediate inner and outer measures. For example, take the union of those points with the points in the upper right quadrant, and then take the intersection of that union with the points not in the lower left quadrant. The resulting set of points will have inner measure $\frac{1}{4}$ and outer measure $\frac{3}{4}$. One can easily produce non-measurable sets of any possible inner and outer measures.

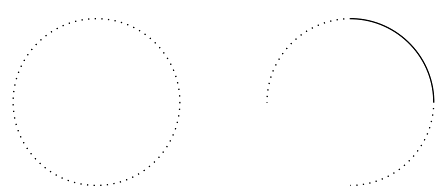

FIGURE 2. Set with inner measure 0 and outer measure 1 (left) and set with inner measure $\frac{1}{4}$ and outer measure $\frac{3}{4}$ (right).

3. Non-measurability and precise credence

So far our discussion of how non-measurability relates to probability has been abstract. We turn now to how non-measurability relates to probabilistic degrees of belief.

Orthodox probabilistic mathematics leaves non-measurable sets out of its fields. Orthodox Bayesianism uses that mathematics, so it also leaves them out of its fields. Consider a spinner that is subjectively fair, in that an agent's credences about where the spinner will land are rotationally symmetric. If the agent's doxastic state is expressed by a probability function, then the proposition that the spinner will land on some particular non-measurable set cannot get *any* probability. Since any assignment of probabilities would violate the symmetry constraint, the constraint requires that no probability be assigned.

One might take such results to suggest that spinners cannot be subjectively fair. While it would be strange to disallow rotationally symmetric credences, perhaps we could learn to live with such strangeness. But Ulam's result (which we mentioned earlier) shows that non-measurability is not so easily avoided. Any agent with precise credences for all the sets of landing points must assign credence 1 to some countable set (still assuming the continuum hypothesis, to be revisited later). Such credences are not only inconsistent with the spinner being subjectively fair, they are also inconsistent with the spinner being subjectively unfair in any natural way (such as any continuous probability distribution). It's one thing to think that rotational symmetry is impossible, but it would be bizarre to require thinkers to put all their eggs in countably many baskets when uncountably many baskets are available. So we have to deal with non-measurability somehow.

Credence gaps do not do justice to the structure underlying credal states—non-trivial bounds provided by inner and outer measures. These allow us to make important distinctions among non-measurable sets. For example, while N may have inner measure 0 and outer measure 1, N' may have inner measure 0.6 and outer measure 0.7, and N'' may have inner measure 0 and outer measure 0.00000000000001. These distinctions are lost if we regard these simply as credence gaps, as orthodoxy would have it.

Moreover, trouble is liable to spread due to correlations between propositions. Suppose the agent is certain that a light bulb is causally connected to the subjectively fair spinner such that it will turn on just in case the spinner lands in some particular non-measurable set N. Then the agent cannot assign a credence to the proposition that the light will turn on. If she did, she would also have to assign it to the proposition that the spinner lands on a point in N, and she cannot have any credence in that proposition. Now suppose the agent merely assigns low (but non-zero) credence to a causal connection between a non-measurable set of points on a spinner and the light—chances are it's an ordinary light connected to an ordinary switch, but it just might be a light that's triggered by the spinner landing in N.[4] Even then, the agent cannot assign a credence to the light's turning on, as that

[4]A reader worried that a rational agent can be certain that there won't be any such causal link, since 'any such causal process must be infinitely precise'. There are natural senses in which mundane physical processes are infinitely precise; the velocity of a point-particle may be as sensitive as you like to its acceleration. But in any case there needn't be any sort of delicate *physical* mechanism connecting the spinner to the light. We can instead imagine God seeing whether or not the spinner landed on a point in some particular non-measurable set and accordingly

credence could not be real-valued. If there is trouble with even a few propositions, that trouble could prove hard to contain.

Orthodox Bayesianism conflates important distinctions—it's measurability or bust. It represents an agent's doxastic attitude to a proposition with her credence in it; if it doesn't appear in her algebra, she simply has no doxastic attitude to it. To be sure, it's fine if the algebra excludes some propositions—one need not think about everything. But if an agent wants to assign a credence to the proposition that the spinner will land on some non-measurable set of points she should not be doomed to failure. And if she thinks that a light may be causally connected to some non-measurable set of points and wants to assign a credence to the proposition that the light will turn on she should not be doomed to failure. An epistemology that abjures such agents is too restrictive. We should welcome a way for Bayesianism to do better.

4. **Imprecision to the rescue**

There is a way. Heterodox Bayesianism takes agents to have *imprecise* credences. Propositions need not get a probability, but may instead be assigned a range of probabilities. Thus we may let a proposition's inner measure correspond to the lower bound for the probability of that proposition and the proposition's outer measure correspond to the upper bound.

To be sure, the idea of applying imprecise probabilities to non-measurable sets has been gestured at before, such as in the title—but not the content—of Good (1966). Binmore states the idea explicitly:

> After the Banach-Tarski paradox, it won't be surprising that we can't say much about non-measurable events in Kolmogorov's theory. But it is possible to assign an upper probability and a lower probability to any non-measurable event E by identifying these quantities with the outer and inner measure of E calculated from a given probability measure p. (Binmore 2009, p. 88)

But this short passage is all that Binmore says on this point (his main concerns lie elsewhere). More generally, the idea of using imprecise credences to deal with non-measurable sets has remained rudimentary—its motivations unarticulated, its details unexplored, its limitations unrecognized, and its relationship to the broader literature undiscussed. We will develop this idea more fully.

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either turning the light on or not. Such scenarios may be highly improbable, but it seems an overreaction to require all rational agents to rule them out entirely.

Given that the spinner is fair, it would seem that the credence that the spinner will land on some set of points should correspond to the length of that set relative to the length of the whole circle. And although non-measurable sets do not have lengths in a standard sense, their inner and outer measures do place something like bounds for the sets' lengths. A non-measurable set is certainly no shorter than any of its subsets, and certainly no longer than any of its supersets. This is well represented by an imprecise credence whose range corresponds to the inner and outer measures of a non-measurable set of points. By contrast, a credence gap reflects none of this structure. Yet an agent with only a precise probability function is consigned to have such a gap. Hence, Bayesianism requires imprecise credences in its representational repertoire.

Our thesis is only that imprecise credences are rationally permissible, not that they are required. We don't presuppose that rationality requires anything more than coherence. And there is, of course, nothing incoherent about a precise credal state; agents can put all their eggs in countably many baskets. However, given a more exacting theory of rationality it could easily be irrational to do so. Given $\aleph_1$ epistemically possible worlds, such a theory might deem it irrational to have non-zero credence in any particular world. Ulam's result would then entail that some sets of worlds cannot receive precise probabilities, and imprecise credences might well be required for such sets.

A simple version of our thesis would have an agent's precise credences determine the appropriate values for her imprecise credences. Even an agent with no precise credence in p is guaranteed to have precise credences in some propositions which entail p and some which are entailed by p.[5] On this simple version of our thesis, if the greatest lower bound of her credences in the entailing propositions is x and the least upper bound of the entailed propositions is y, then the appropriate range for her imprecise credence in p is $[x, y]$. This thesis has the advantage that the dynamics of imprecise credences are fairly straightforward, as the rule for updating precise credences will automatically specify the rule for updating imprecise credences.

There are potential issues with this simple version, however, as it makes coherence requirements quite sensitive to the specifics of an agent's algebra. Think back to the set of points in Figure 2, the one with inner measure $\frac{1}{4}$ and outer measure $\frac{3}{4}$. Let the proposition that the spinner lands on a point in that set be p. As we said, an agent who

[5]Remember that a contradiction will entail p and a tautology will be entailed by p.

regards the spinner as subjectively fair should have imprecise credence $[\frac{1}{4}, \frac{3}{4}]$ in p and imprecise credence $[\frac{1}{4}, \frac{3}{4}]$ in $\neg p$. But now consider an agent who has doxastic attitudes towards only a very limited set of propositions: $\{\top, p, \neg p, \bot\}$. According to the simple version, the only imprecise credences that agent would be permitted to have regarding p and $\neg p$ would be [0, 1].

An alternative version of our thesis that avoids this consequence would only require that it be possible to enrich an agent's algebra in such a way that the ranges of her imprecise credences correspond to the bounds of her precise credences.

On the simple version of our thesis rational imprecise credences supervene on rational precise credences, whereas on the alternative version of our thesis they do not. We are agnostic regarding which version is correct; our task here is only to argue for the rational permissibility of some imprecise credences.

5. **Imprecise chances and the principal principle**

Various versions of the Principal Principle (Lewis 1980) codify a correspondence between chances and rational credences—so long as the chances are measurable. When one is certain that the chance of a proposition X is p (and has no inadmissible information), one's credence in X should be p. Stated as a constraint on rational conditional credences (and suppressing the admissibility clause), we have the schema:

Principal Principle: $Cr(X|Ch(X) = p) = p.$

From this it follows that one's credence in X should equal one's expectation of the chance of X—whenever this expectation exists.[6] The credence measure is thus guided by the chance measure—it's measure for measure.

But many of the issues that non-measurability poses for precise credences apply equally to precise chances. Consider a spinner that is objectively fair, in that chances for where the spinner will land are rotationally symmetric. If the chances must be precise then the proposition that the spinner will land on some particular non-measurable set cannot get any chance. And even if the spinner is not objectively fair, it is still very hard for all the sets of points to have precise chances of being landed on. This would require that there be some countable set of points such that the spinner has chance 1 of landing on one of them.

[6]Assuming that credences are conglomerable, i.e. that unconditional credences fall in the range of conditional credences. For more, see Easwaran (2019).

Since chances will not always have this structure, if chances must be precise then some propositions will not have a chance.[7]

Let us say that a proposition that cannot have a real-valued chance given operative constraints is *chance non-measurable*.[8] What should one do when a known chance function makes a proposition chance non-measurable? And what should one do when one's uncertainty about possible chances means that one has no real-valued expectation of the chance? Standard formulations of the Principal Principle only deal with measurable chances, and so they do not apply. Knowing that a proposition has some measurable chance constrains one to have a corresponding precise credence, yet knowing that a proposition is chance non-measurable imposes no constraints at all—however much one knows about it. But that can't be right. There must be more to the epistemic significance of objective chance than is contained in standard formulations of the Principal Principle.

We may easily generalize the import of chances. Just as the measure of some set of points corresponds to the chance that the spinner will land on one of its points, we can say that a set's inner measure corresponds to the *inner chance* that the spinner will land on one of its points and that a set's outer measure corresponds to the the *outer chance* that the spinner will land on one of its points. We may interpret this as an imprecise chance: it is imprecise over the interval: [inner chance, outer chance]. When the inner chance of a proposition is known, the lower bound of one's credence in that proposition should correspond to it. When the outer chance of a proposition is known, the upper bound of one's credence in that proposition should correspond to it.[9] One's credence is imprecise over the same interval.

We may generalize to cases in which the imprecise chance is not known, taking our lead from the original Principal Principle. It imposed a constraint on a conditional credence—conditional on a specific hypothesis about a precise chance. Now we may allow the hypothesis to be about an imprecise chance:

Generalized Principal Principle: $Cr(X|Ch(X) = [p_i, p_o]) = [p_i, p_o]$.

[7] If an agent knows that each point has chance 0 of being landed on and thus assigns each point credence 0, then Ulam's result dictates that the agent cannot assign every set of points a precise credence.

[8] A proposition that *could* receive a chance might nonetheless *not* receive one. Many of the implications of chance non-measurable propositions hold equally for such propositions.

[9] We leave aside the question of how an agent would come to know that a proposition has some imprecise chance. The specifics are liable to depend on the operative theory of objective chances.

Here, the condition says that the chance is imprecise over the interval $[p_i, p_o]$, where p_i is the inner chance, and p_o is the outer chance. This Generalized Principal Principle constrains our conditional credence in a proposition to coincide with its hypothesized chance, as before; it's just that now that chance may be imprecise. We submit that the generalized principle is plausible in much the same way that the original principle is. Chance acts as an 'expert', in the sense of Gaifman (1988); but even experts can be imprecise. The generalized principle reduces to the original principle when the hypothesized chance is sharp.

Still more generally, when an agent distributes credences over multiple chance hypotheses her credence in any proposition X should span the interval from X's expected inner chance to X's expected outer chance. When X has a measurable chance, the standard constraints of the Principal Principle follow. But when X is chance non-measurable, this generalization of the Principal Principle does not fall silent. Instead, it mandates an imprecise credence.

6. **Interpreting imprecise credences**

The most common interpretation of imprecise credences ascribes sets of probability functions to agents, a sort of credal committee.[10] According to such interpretations the range of values an agent assigns to a proposition is determined by the values assigned by those probability functions. Defending this view, Bas van Fraassen writes,

> Our subjective probabilities are not usually very precise. Rain does seem more likely to me than not, but does it seem more than π times as likely as not? That may not be an answerable question. The standard remedy, as elaborated in the literature by Isaac Levi, Richard Jeffrey, Patrick Suppes, among others, is to represent opinion by a class of probability functions rather than a single one... To allow for vagueness, a person's state of opinion can thus be represented by a class of probability functions—call it the representor. His or her (vague) degrees of belief are meant to be captured by what the members of that class have in common. (van Fraassen 2006, Pp. 483-484)

Relatedly, Sarah Moss writes that 'we should take seriously the suggestion to think of an agent with imprecise credences as if she had a mental committee of agents with precise credences. Such an agent should act in whatever ways a committee should' (Moss 2009, p.68).

[10]For more see Rinard (2017).

While our reasoning is friendly towards imprecise credences, it is unfriendly towards the standard way of interpreting them. We reject the idea of a credal committee. In the cases we have in mind, rational agents need not be vague nor somehow split among precise credences. Rather, they pointedly reject any precise credence. Vitali's theorem shows that it is impossible for precise credences to respect the symmetries of a perfectly fair spinner. Any precise numerical assignment within the imprecise range would violate rotational symmetry. It would be dubious for an agent who considers a spinner to be fair to have a credal committee whose members all treat the spinner as biased. And Ulam's result creates even more severe problems for the credal committee approach. Consider an agent who assigns each point probability 0 of being landed on and thus has some imprecise credences regarding some sets of points. The members of that agent's credal committee would also assign probability 0 to each point, and would assign precise probabilities to all the sets of points. But Ulam's result shows that this is impossible—any precise numerical assignment which assigned 0 to each point and some number to each set of points must be probabilistically incoherent. Precise probability functions—even en masse—can't do what needs doing. The imprecise credences we advocate are nothing like sets of precise probabilities.

What, then, is an imprecise credence? There are various theories (just as there are for precise credences). One example is Sarah Moss' (2014) view that credences are beliefs with certain probabilistic contents. According to this view, having credence x in a proposition means believing that it has probability x. This is, in effect, believing that it has both inner probability and outer probability x. It would be no more problematic for imprecise credence $[x, y]$ in a proposition to amount to the belief that it has inner probability x and outer probability y.[11] For those of more operationalist inclinations, there's also Anna Mahtani's (2016) view that an imprecise credence represents a particular pattern of unstable betting-behavior. We take no stand on what theory of imprecise credence is correct, maintaining only that some such theory is.

[11]Though this theory of imprecise credences is inspired by Moss, we note that one cannot represent such probabilistic beliefs in the way Moss wants—namely, as believing a set of (precise) probability spaces. The extent to which the particulars of Moss' framework can be adapted to these purposes remains to be seen. For more, see Moss (2016).

7. **Imprecision and its critics**

Many seemingly powerful arguments have been levelled against the rational permissibility of imprecise probabilities. But probabilistic imprecision brought about by non-measurable propositions casts these arguments in new light. Not only do non-measurable propositions provide a strong argument for probabilistic imprecision, they undermine seemingly strong arguments against it. Non-measurability even shows that by the very lights that have been used to argue against imprecise credences, imprecise credences are better than precise credences. We offer rejoinders to arguments made by Adam Elga, Roger White, and Miriam Schoenfield.

Elga:

Adam Elga (2010) argues that imprecise probabilities are bad for decision theory. Elga's argument centers around a sequence of two bets about some proposition H: Bet A would cause a loss of \$10 if H is true and a gain of \$15 if H is false. Bet B would cause a gain of \$15 if H is true and a loss of \$10 if H is false. Elga notes that an agent with a precise credence in H is guaranteed to take at least one of the two bets, and argues that any agent is rationally required to do so. But Elga shows that the most natural decision theories for imprecise probabilities do not require an agent to take at least one of the two bets. Thus Elga concludes, 'How do unsharp probabilities constrain rational action? ... [T]here is no good answer to that question' (Elga 2010, p. 10). Admittedly, further decision theories for agents with imprecise probabilities have been developed, some of which would require any agent to accept at least one of the two bets.[12] Nonetheless, these imprecise decision theories are subject to other criticisms; it may still seem reasonable to consider any imprecise decision theory to be worse than traditional, precise decision theory.

Elga's argument presupposes that the alternative to an imprecise probability assignment and imprecise decision theory is a precise probability assignment and precise decision theory. But the existence of non-measurable propositions shows that this is not so. In cases with constraints that engender non-measurability, the alternative to an imprecise probability assignment and imprecise decision theory is no probability assignment and no decision theory. Traditional decision theory simply breaks down when it comes to propositions towards which an agent has no credence. It's not hard for an imprecise decision theory to be better than that.

[12]See Sahlin and Weirich (2014) and Rinard (2015) for more.

The mathematics of non-measurable sets can provide decision-theoretic guidance. A non-measurable set's inner measure gives a lower bound for its size and its outer measure gives an upper bound. Inner and outer measures thus provide compelling constraints on decision-making. Let N be a non-measurable proposition, let i be its inner measure, and let o be its outer measure. Let P be a proposition with probability i, and let Q be a proposition with probability o. Then a bet on N is worth at least as much as a bet on P, and no more than a bet on Q (given a fixed payoff structure). These verdicts are vindicated by dominance reasoning when the measurable propositions in question are subsets or supersets of N.

We do not advocate any particular decision theory for agents with imprecise probabilities, nor do we need to. We advocate only a minimal constraint for imprecise decision theories, a constraint which all extant imprecise decision theories satisfy.

> **Range Dominance:** Let o_1 and o_2 be the only possible outcomes, with $u(o_1) > u(o_2)$. Let the probability of o_1 conditional on performing act a_1 be $[x_1, y_1]$ and let the probability of o_1 conditional on performing act a_2 be $[x_2, y_2]$. If $x_1 > y_2$, then a_1 is strictly preferred to a_2.

Range Dominance says that when there are only two possible outcomes and one act is unarguably more conducive to the better outcome, prefer that act.[13] This principle gets some guidance out of imprecise credences. Traditional decision theory gets no guidance out of credal gaps.

We thus turn Elga's argument on its head. A restriction to precise credences is bad for decision theory, thus demonstrating that imprecise probabilities are rationally permissible. A wider range of decision problems are solvable if imprecise credences are allowed.

White:

Roger White (2009) argues that imprecise probabilities lead to unreasonable patterns of reasoning. White shows that it's possible to force your sharp 50/50 credences about how an avowedly fair coin will land to dilate, and even to dilate maximally.[14] All we need is a fair coin, some stickers, a proposition p about which your imprecise probability ranges over the entire interval from 0 to 1, and someone who knows whether p is true or not.

[13] Range Dominance can easily be generalized to countable outcome spaces.

[14] For prior treatments of dilation, see Seidenfeld and Wasserman (2003) and Walley (2001).

Here's how: Get two small stickers, one of which is marked 'p' and one of which is marked '$\neg p$'. Have the person who knows whether p or $\neg p$ is true put the sticker corresponding to the true proposition onto the Heads side of the coin and the sticker corresponding to the false proposition onto the Tails side of the coin. The stickers must hide the faces of the coin, but not affect its fairness. Then flip the coin.

Before the coin is flipped, you have a sharp .5 credence about whether the coin will land Heads or Tails. But then the coin is flipped. You can see whether the coin landed with 'p' or '$\neg p$' facing up. Suppose (without loss of generality) that the coin landed with 'p' facing up. You then know that the coin landed Heads if and only if p is true and that the coin landed Tails if and only if p is false. Given a standard theory of updating, your credence about how the coin landed must dilate, spanning the full interval from 0 to 1.

This consequence of the imprecise probability framework may seem disquieting, and White's case is considered a major challenge for imprecise probabilities.[15] There are, we think, two intuitions that account for the disquiet.[16]

(1) Your credences about how the coin landed should not change at all.
(2) Your credences about how the coin landed should not be guaranteed to change in some particular way.

We think that (1) accounts for the bulk of the disquiet. People seem to be troubled that your credences about how the coin landed change at all; they are less troubled by the predictability of the change.

We maintain, however, that (1) is false. It's entirely proper for credences about how a fair coin landed to be affected by circumstances like White's. Suppose that instead of having imprecise credence spanning the full [0, 1] interval regarding p you had sharp credence .7. Then observing that the coin landed with the p sticker up should lead you to adopt credence .7 that the coin landed Heads and .3 that it landed Tails. After you observe that the coin landed with the p sticker up, your credence that the coin landed Heads must match your initial credence that p is true. This form of reasoning is even more obvious if you had a sharp credence of 1 in p—then upon seeing the coin land with 'p' facing up, you should have credence 1 that the coin landed Heads.

[15]For dissent, see Joyce (2010).
[16]Our speculation is based on conversations with friends and colleagues.

The only case in which your credence that the coin landed Heads remains unchanged is the case in which your initial credence in p is .5, and that's only because then your initial credence about p happens to correspond to your initial credence that the coin would land Heads. Clearly your credence that the coin landed Heads may be affected by the observation that the coin landed with the p sticker up. This is the case whether or not your probability for p is imprecise.

If White's case is to pose a problem for imprecise probabilities, it must be because the dilation occasioned by the coin toss is predictable. If you had sharp credence .7 in p, then your credence that the coin landed Heads might wind up being either .7 or .3 depending on how the coin lands. But if your imprecise credence in p spans the full [0, 1] interval, then however the coin lands you are sure to adopt imprecise credence [0, 1]. This is admittedly strange. The (potential) predictability of dilation is a distinctive consequence of imprecise probabilities, and thus may pose a distinctive problem.

But does it? Why should it be unreasonable to know that some particular epistemic change is going to occur? One might think that some sort of Reflection principle requires that any time you are certain that you are going to adopt some epistemic state in the future, you adopt it now. There are, however, myriad well-known counterexamples to such Reflection principles: irrationality, evidence-loss, centered propositions, a lack of introspective access, and so on.[17] Why not just allow that imprecise probabilities pose another such counterexample? The conditions under which Reflection holds are well-understood; R.A. Briggs (2009) shows the conditions under which Reflection is entailed by probabilistic coherence. Those conditions are not met by some agents with imprecise probabilities, but it is equally true that they are not met by some agents with precise probabilities. Thus, failures of Reflection give no particular grounds for preferring precise credences to imprecise credences.

One might think that one's future self is going to be better-informed than one's present self, and thus that one should defer to that future self's credences. White writes, '[A]s long as you know that you will be epistemically fine … you should trust your future judgment and match your current attitude to it'(White 2009, p.178). But White's line of reasoning cannot validate precise probabilities, as they too require a violation of Reflection. A precise agent's credences about the coin predictably change from .5 to each outcome to a *gap* for each outcome.

[17]See Christensen (1991), Talbott (1991), Arntzenius (2003), Maher (1992), Bovens (1995), and Meacham (2008).

Before the coin is tossed a precise agent has a sharp .5 credence in Heads and in Tails. But after the coin is tossed the precise agent cannot have any credence about how the coin landed—the domain of propositions to which she assigns credences must contract. She can foresee that she will not have any epistemic attitudes about how the coin landed in the future, so a Reflection principle would mandate that she not have any credences *now* about how the coin will land. Moreover, this predictable contraction of the domain is more radical and more troublesome a change than would be required within the imprecise probability framework. To be sure, dilation may be puzzling, especially when it is anticipated. But the loss of a probability assignment altogether is more puzzling, especially when it is anticipated. White does not avoid violating Reflection by insisting on sharp probabilities; he violates Reflection—in a particularly egregious way—because he insists on sharp probabilities. Reflection principles provide no support for precise probabilities over imprecise probabilities, and they might well provide support for imprecise probabilities over precise probabilities.

Schoenfield:

Miriam Schoenfield (2017) argues that accuracy-theoretic considerations suggest that imprecise probabilities cannot be rationally required. Schoenfield notes that the normal accuracy-theoretic framework is structured so that coherent, precise probabilities cannot be accuracy-dominated. It clearly follows that no imprecise probabilities can accuracy-dominate precise probabilities. Schoenfield argues that—given natural assumptions about the structure of scoring rules—the accuracy non-domination of precise probabilities will entail that for any imprecise probability there is a precise probability that is never less accurate than it. Schoenfield thus argues that accuracy-theoretic considerations never mandate imprecise probability. Schoenfield's argument centers on a case she dubs MYSTERY-COIN. She writes,

> [I]f we take an accuracy-centered approach to epistemology, the imprecise credal state recommended in MYSTERY-COIN (call it **i**) *can't* be rationally required because there is a precise probability function, (call it **p**), defined over H/T that is no less accurate than **i** in every world (Schoenfield 2017, p.678).

Although Schoenfield's argumentation focuses on one case, her reasoning can be generalized.[18]

[18]See Berger and Das (2019).

The accuracy-theoretic framework is, however, unsuitable for dealing with non-measurability. First, the spinner case that motivated our argument involves an infinite outcome space, and the accuracy-theoretic framework faces substantial complications in infinite domains.[19] Second, given the relevant constraints, agents with precise probabilities cannot assign any probabilities whatsoever to some propositions. So accuracy-theoretic comparisons between precise probabilities and imprecise probabilities are beside the point; one would need an accuracy-theoretic comparison between a gap and an imprecise probability. We know of no accuracy-theoretic arguments that gaps should be preferred to imprecise probabilities.

8. **Questioning the intelligibility of non-measurable sets**

Some may worry about the intelligibility of specific non-measurable sets. The worry goes something like this: 'The propositions to which highly idealized agents assign imprecise credence are propositions that limited agents like us cannot grasp. So while Bayesian superbeings might need imprecise credences, for us they are irrelevant.'

We think this worry is easily answered. First, it's tendentious to allege that a proposition that concerns a spinner landing on a specific non-measurable set of points is unintelligible to agents like us. Even if such a singular proposition is somehow too mathematically complex to entertain under a mathematical guise, it can still be entertained under some other guise. If God tells you that he's picked out a non-measurable set of points with inner measure .25 and outer measure .75 and named it 'Bob', then you should get cognitive access to the proposition that the spinner lands on Bob via a standard Kripkean chain of communication.[20] Furthermore, the guise under which this proposition is entertained should not affect whether it forces an imprecise credence in this case.[21] Thus even the unavailability of a mathematical guise would not insulate an agent from the proposition. Second, even if such a proposition is unintelligible, there are plenty of propositions left that force imprecise credences. Existential propositions do so, and they are unproblematic. Grant for the sake of argument that an agent cannot have positive credence that that a light is connected to a *specific* set of

[19]If all propositions contribute equally to inaccuracy then overall inaccuracy is liable to be infinite. See Joyce (1998), Joyce (2009), and Pettigrew (2016) for more on these issues, and Easwaran (2013) and Kelley (MS) for technical approaches to them.

[20]See Kripke (1980).

[21]For some peculiar cases in which guises make an epistemological difference see Hawthorne (2002).

points with chance inner measure .25 and outer measure .75; still, the agent can have positive credence that the light is connected to *some* set of points with chance inner measure .25 and outer measure .75.

Moreover, even if agents can only entertain finitely many thoughts, some of those thoughts can still be about imprecise chances—the thought that the chance of some proposition is [.25, .75] is not especially hard to entertain. And once one grants that agents may assign imprecise credences in such circumstances, it is hard for someone with permissive sensibilities to deny that they may be assigned more broadly.

9. Ways out: the axiom of choice, symmetry, countable additivity, and the continuum hypothesis

Our arguments for the legitimacy of imprecise credences rely on some mathematical claims and on some epistemological claims. By denying those claims our arguments can be resisted. (Our arguments for the legitimacy of imprecise chances similarly rely on some mathematical claims and some claims about the nature of chances, and may be resisted through analogous denials.)

Here are some ways to maintain that precise agents are not (as we have maintained) menaced by credal gaps, but instead that they can easily have precise credences in any proposition they like. There are four crucial notions: the axiom of choice, symmetry, countable additivity, and the continuum hypothesis. Resisting our reasoning requires either rejecting the axiom of choice, rejecting both symmetry and countable additivity, or rejecting both symmetry and the continuum hypothesis.

The Axiom of Choice:

Proofs of the existence of non-measurable sets rely on the axiom of choice. This axiom says that for any set of non-empty sets there is a function that picks out one element from each set. Denying the axiom of choice makes it possible (though not obligatory[22]) to deny the existence of non-measurable sets, and thus provides a path for resisting our argument for imprecise probabilities.[23] One might, for example, adopt the axiom of determinacy, which entails that all sets of real numbers are measurable.[24]

[22]Note that the Hahn-Banach theorem entails the existence of non-measurable sets and can be proved with a slightly weaker axiom. See Foreman and Wehrung (1991).

[23]For more, see Solovay (1970), Shelah (1984), and Shelah and Woodin (1990).

[24]For more on the axiom of determinacy see Mycielski and Steinhaus (1962).

Symmetry:

Recall that when mathematicians say that a set is non-measurable, they do not mean that it is impossible to assign it a measure no matter what, but only that it is impossible to assign it a measure consistent with some natural constraints. If one maintained that these constraints, though natural enough in mathematics, were unacceptable in epistemology then precise credences could be assigned unproblematically.

Our Vitali-inspired proof of non-measurable sets involved a subjectively fair spinner, in that the agent's credences about where the spinner will land have rotational symmetry. But other, even more striking sorts of non-measurability emerge from other sorts of symmetry.

Consider the fair selection of a point from a three dimensional region of space, in that an agent's credences about the point's location have rotational and translational symmetry. In 1914, Felix Hausdorff showed that if a countable set of points is removed from a sphere, then the remaining points can be divided into three disjoint subsets A, B, and C such that A, B, C, and $B \cup C$ are all congruent. In 1924, Stefan Banach and Alfred Tarski showed that a sphere can be decomposed into 5 disjoint sets of points which, rigidly transformed, recompose into two spheres of equal volume to the initial sphere. The Hausdorff and Banach-Tarski demonstrations of non-measurability only require finite additivity, so given the axiom of choice and the symmetry constraint it will be impossible to assign precise credences to all propositions.

Rejecting the symmetry constraint for credences could limit the epistemological significance of Vitali's proof and the Hausdorff and Banach-Tarski paradoxes. But one would need to disallow not only perfect credal symmetry, but anything remotely approaching it. The Hausdorff paradox demonstrates that for any precise probability function and any $\epsilon > 0$, there must be disjoint, congruent sets of points such that one set has probability less than $0 + \epsilon$ and another has probability greater than 1 - ϵ. Possibilities that would be treated identically by an agent with a symmetrical sensibility would have to range from virtually impossible to virtually certain. Such a rejection of symmetry would prevent the Vitali, Hausdorff, and Banach-Tarski sorts of non-measurability.

Countable Additivity:

Countable additivity can lead to non-measurability even in the absence of any symmetry constraints. Suppose that there is a spinner that can land on uncountably infinitely many points and that each point has probability 0 of being landed on. Assuming the continuum hypothesis, Ulam's result shows that if credences are precise and countably additive then some sets of points do not receive a credence. And it won't help to have a few points have minute probabilities. So long as the credences are only countably additive (as opposed to additive at every cardinality) it will be impossible to assign credences to all sets of points.

But if credences do not have to be countably additive in the first place, then no problem emerges.[25] Adopting a non-standard measure (in which infinitesimal probabilities may be assigned) is much the same as denying countable additivity, as it means that the standard components of probabilities need not be countably additive.[26] Such rejections of countable additivity would prevent the Vitali and Ulam sorts of non-measurability.

The Continuum Hypothesis:

Ulam's result applies to sets of cardinality $\aleph_1$. If the continuum has cardinality $\aleph_1$ then the application of that result is straightforward. But what if the continuum has cardinality greater than $\aleph_1$?[27]

Even if the continuum hypothesis is false, there are many ways for Ulam's result to militate for imprecise credences. First, Ulam generalized his result to all cardinalities that are not weakly inaccessible, so it applies to many cardinalities other than $\aleph_1$. Even if the continuum hypothesis is false the result might apply directly to the cardinality of the continuum.[28] Second, even if Ulam's result doesn't apply to the cardinality of the continuum, an agent with non-zero credence that Ulam's result applies to the continuum might be obliged to structure the relevant portion of their credences as though it it did apply to the continuum. We freely grant that modeling such uncertainty about the structure of set theory is highly fraught, but think the possibility is worth flagging. Third, if the continuum hypothesis is false it means that the continuum's cardinality is greater than $\aleph_1$, which means that there will be subsets of the continuum to which Ulam's result does apply.

[25]To avoid problems, in appropriate circumstances violations of countable additivity would need to be required, not merely permitted.

[26]Strictly speaking, a non-standard measure would not respect countable additivity, as the notion of the countable addition of non-standard numbers is undefined. For more about the mathematics of infinitesimals see Robinson (1961). For more on non-standard measures see Bernstein and Wattenberg (1969) and Luxemburg (1973).

[27]The continuum hypothesis is provably independent of Zermelo-Fraenkel set theory with the axiom of choice. For more, see Gödel (1940) and Cohen (1963).

[28]There are models of ZFC + ¬CH in which the continuum is not weakly inaccessible. For a thorough exploration of relevant issues, see Hoek (Forthcoming).

So God could make $\aleph_1$ points glow and then tell you that he's picked one of those glowing points. Your credences about which point God picked would then be constrained by Ulam's result. Suppose that your credences over the points are not extremely biased to just a countable set: you have no reason for privileging any such set in such a drastic way. Then your credences are not defined on some subsets—they are non-measurable sets for you. Granted, if one denied the continuum hypothesis, denied that Ulam's results applied to the continuum, denied that uncertainty about the continuum hypothesis had pertinent implications, and either denied the possibility of God picking a point out of $\aleph_1$ points or denied the legitimacy of abjuring extremely asymmetric credences over the points, then one could then resist the pertinence of Ulam's result even if one accepted countable additivity.

We have mentioned four demonstrations of non-measurable sets: by Vitali, Hausdorff, Banach-Tarski, and Ulam. By rejecting the axiom of choice and accepting the axiom of determinacy one avoids all four sorts of non-measurability. Given the axiom of choice, avoiding all four sorts of non-measurability requires either rejecting both symmetry and countable additivity or rejecting both symmetry and the continuum hypothesis.

The axiom of choice, the relevant symmetry constraints[29], and countable additivity are all widely accepted. We are disinclined to reject such orthodoxies simply because they turn out to lend support to heterodox philosophical hypotheses. We are more agnostic about the continuum hypothesis, but think it possible to be confronted with a sample space of cardinality $\aleph_1$ regardless. We want to follow the arguments where they lead—in this case, to imprecise credences.

10. **Conclusion**

Precise credences aren't as flexible as one might have supposed; the phenomenon of non-measurability poses problems. Though there are—as we detailed—various ways to maintain that probabilistic epistemology can make do with only precise credences, these come with serious costs. The emerging unattractiveness of a restriction to precise credences supports the central thesis of this paper: that imprecise credences are rationally permissible.[30]

[29]Not that credences must be symmetric, but that they may be.

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