

Probabilities Cannot Be Rationally Neglected

Yoaav Isaacs

Abstract

In response to Smith 2014, I argue that probabilities cannot be rationally neglected. I show that Smith's proposal for ignoring low probability outcomes must, on pain of violating dominance reasoning, license taking arbitrarily much risk for arbitrarily little reward.

Nicholas Smith argues that the value that an ideally rational agent places on a gamble need not equal that gamble's expected value.¹ Instead, Smith argues that outcomes of sufficiently low probability may fail to influence a gamble's valuation—that some probabilities are rationally negligible.

Rationally Negligible Probabilities (RNP): For any lottery featuring in any decision problem faced by any agent, there is an $\epsilon > 0$ such that the agent need not consider outcomes of that lottery of probability less than ϵ in coming to a fully rational decision (Smith 2014).²

RNP allows problematic infinite gambles (like the St. Petersburg game or the Pasadena game) to be reduced to unproblematic finite gambles. RNP's contextual sensitivity makes it a flexible principle; the value of ϵ can change from lottery to lottery. Yet despite that flexibility RNP has untenable implications.

Smith's employment of RNP is not totally unsystematic. He considers various constraints for the values that ϵ can take, and endorses the weakest of them. This principle is

Weak Consistency: If a single decision problem d involves two lotteries L_1 and L_2 over the same set of outcomes, then if one truncates L_1 at ϵ for purposes of addressing d , one must also truncate L_2 at ϵ for purposes of addressing d .

¹When such a value exists.

²According to Smith, the probabilities of the remaining outcomes should be renormalized. The value of the lottery would then be the expected value according to these renormalized probabilities.

But a troubling result easily follows from Weak Consistency: RNP must license taking arbitrarily much risk for arbitrarily little reward.

Let us consider a fair coin which will be tossed until it lands Heads.³ We will define some lotteries over that set of outcomes, and an agent will choose whichever of them he likes best. One available lottery is Status Quo, which pays off \$0 however the coin lands.⁴ Another available lottery is Bad News, which is a bit more complicated.

Bad News: The payoff of the game is $-\$(2^{3n-2})$, where n = the total number of tosses performed.⁵

Bad News bears some resemblance to the classic St. Petersburg game. Each gamble guarantees a finite payoff, yet the expected value of each gamble is infinite. But in Bad News, unlike in the St. Petersburg game, the payoffs are negative and the terms make increasingly significant contributions to expected utility.⁶ Also available are lotteries of the form Penny or Doom _{n} , for all $n > 0$. These pay one penny so long as the coin is not flipped exactly n times, and pay $-\$(2^{3n-2})$ if the coin is flipped exactly n times.

Smith must permit an agent to prefer infinitely many versions of Penny or Doom to Status Quo.⁷ Smith must permit an agent to evaluate infinitely many versions of Penny or Doom as being worth a penny, despite the fact that they have enormous (and growing) negative expected payouts. The proof is simple. RNP mandates that some of the outcomes in Bad News may be ignored.⁸ Weak Consistency mandates that the same outcomes be negligible in all of the lotteries.⁹ Thus outcomes of sufficiently low probability will be negligible in all of the lotteries.

³The coin may continue landing Tails forever, but this unusual scenario is of no moment for what follows.

⁴For simplicity, I will make the standard assumption that an agent's utilities over money are linear. This simplification could be dispensed with so long as the agent's utilities remained unbounded.

⁵If the coin never lands Heads the payoff of the game is 0.

⁶The St. Petersburg game can be defined over the same setup, but with payoffs of $\$2^n$. Ordered according to their probabilities, the outcomes of each lottery have probabilities of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, etc. In Bad News, these outcomes have values of $-\$2, -\$16, -\$128, -\1024 , etc., thus making their contributions to expected utility $-\$1, -\$4, -\$16, -\64 , etc. In the St. Petersburg game, these outcomes have values of $\$2, \$4, \$8, \16 , etc., thus making their contributions to expected utility $\$1, \$1, \$1, \1 , etc.

⁷A version of Penny or Doom being a lottery of the form Penny or Doom _{n} for some $n > 0$.

⁸The probabilities of the outcomes in Bad News approach 0 asymptotically. Therefore whatever the value for ϵ set for Bad News, some (indeed, infinitely many) of its outcomes will have probabilities less than ϵ .

⁹Weak Consistency implies that in a decision problem involving any number of lotteries (even infinitely many) if an agent truncates one of the lotteries at ϵ that agent must truncate all of the lotteries at ϵ . Repeated applications of the stated principle easily secure this result.

Therefore infinitely many versions of Penny or Doom will be such that their negative outcomes may be ignored, thus making it rational to value them at a penny. And a penny is better than nothing.

So lotteries with a very high probability of producing a penny and a very low probability of producing a calamitous loss may be rationally evaluated as though they were sure to produce the penny. The arbitrarily calamitous risk may rationally be ignored. But it seems deeply wrong to ignore the obviously salient possibility of calamitous loss.

Given Weak Consistency, RNP must license taking arbitrarily much risk for arbitrarily little reward. But this untenable result is hard to avoid even if one gives up on Weak Consistency. On pain of violating dominance reasoning, RNP must still license taking arbitrarily much risk for arbitrarily little reward. And Smith cannot easily give up on dominance reasoning. Not only does Smith explicitly endorse dominance reasoning, he represents RNP's consistency with dominance reasoning as one of its major strengths. "One important upshot of the approach will be that it allows us to *retain* dominance" (Smith 2014).

Let us again consider an agent who must choose between Status Quo, Bad News, and all versions of Penny or Doom. Given only dominance reasoning Smith must still permit an agent to prefer infinitely many versions of Penny or Doom to Status Quo. Here's the proof: Again, RNP mandates that only finitely many of the terms in Bad News cannot be ignored, thus making it possible to assign Bad News a finitely negative value. But the expected values of the versions of Penny or Doom decrease without limit. Therefore, if the versions of Penny or Doom must be valued according to their expectations, Bad News will be preferable to infinitely many of them. But any version of Penny or Doom is, by dominance, preferable to Bad News—each version of Penny or Doom is just Bad News with all but one of its bad outcomes replaced by a nice penny. Therefore infinitely many versions of Penny or Doom need not be valued according to their expectations. These versions of Penny or Doom must have bad outcomes which may be ignored, thus these versions of Penny or Doom may be rationally valued at a penny. And once again, a penny is better than nothing.

RNP allows an agent evaluating the versions of Penny or Doom to ignore the very low probabilities of the very, very bad outcomes. RNP licenses looking at such gambles and thinking, "Ooh! Free penny!". But the versions of Penny or Doom do not offer a free penny. They offer pennies that come with terrible risk. Such gambles should not be taken blithely. Such gambles should not be taken at all.¹⁰

RNP is an apparently flexible principle, yet it is subject to serious constraint. On pain of violating dominance reasoning, RNP must license taking arbitrarily much risk for arbitrarily little reward. RNP makes for a tractable decision theory,

¹⁰This judgement is widely, but not universally shared. See Alexander 2011 for a dissenting view.

but not a sound one. Probabilities cannot be rationally neglected.¹¹

References

Alexander, J. McKenzie, 2011. ‘Expectations and Choiceworthiness’ *Mind*, 120 (479), pp. 803-817.

Smith, Nicholas J. J., 2014. ‘Is Evaluative Compositionality a Requirement of Rationality?’ *Mind*, 123 (490), pp. 457-502.

¹¹For helpful feedback, I thank Alan Hajek, Wolfgang Schwarz, Adam Elga, and an anonymous referee. This publication was made possible by support from ANU.